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Main Variance components in GPS measurements

Abstract. The author's studies of the structure of GPS measurements errors have resulted in a conclusion that the variance of the GPS measurements is contributed by three main variance components which are due to: 1) tropo-iono variation, 2) quasi-stationary atmospheric refraction blocks and 3) purely random variations in the GPS measurements.

Keywords: GPS; Main Variance Components;Quasi-Stationary Refraction Blocks; Tropo-Iono Variation.

1. Introduction

In [12], [17] Perović characterised the GPS measurements as random processes. So the total error was treated as a process in time with *three basic structural groups of errors*. The first one is due to *tropospheric and ionospheric variations* in time, α, the second is due to *the influence of quasi-stationary refraction blocks*, β, and the third one is due to *purely random errors*– due to random process noise, ε. The author's studies are in favour of approximate constancy of the errors α within a relatively long time interval – from 3 to 6 h, i. e. this interval is $3\text{ h} \lesssim I_\alpha \lesssim 6\text{ h}$, where as the β errors are constant within a comparatively short time interval from 5 min to 2 h, i. e. $5\text{ min} \lesssim I_\beta \lesssim 2\text{ h}$. Since the interval for GPS measurements registration is very short, usually not more than 30 sec, the random errors ε are nested in the random errors β, the random errors β – in the random errors α. So the author discovered that the GPS measurements errors can be described by means of a *two-factor hierarchical model with random effects*. Therefore, for the purpose of the VC analysis he has examined and applied the PERG2FH[2] ANOVA GPS measurements method.

2. Two factor hierarchical modelof GPS measurements

The equation of the model of GPS measurements and of their errors, with the two-factor hierarchical (nested) classification structure with random effects, in succession is (Perović [12], [17]):

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[2] This is an acronym for the author's initials: **PER**ović **G**ligorije's **2**-**F**actor's **H**ierarchical analysis (of variance components) with random effects.

$$\left.\begin{array}{ll}(a)\ Y_{ijk}=\mu+\alpha_i+\beta_{ij}+\varepsilon_{ijk} & -for\ measurements\\ (b)\ \Delta_{ijk}=\alpha_i+\beta_{ij}+\varepsilon_{ijk} & -for\ errors\end{array}\right\};\tag{1}$$

$$i=1,2,\cdots,a\geq 2,\quad j=1,2,\cdots,b\geq 2,\quad k=1,2,\cdots,n\geq 2,\tag{2}$$

and with regard to the said above about the effects their distribution will be normal, so that the *stochastic model* will be:

$$\alpha_i\sim N[0,\sigma_\alpha^2],\ \forall i;\ \beta_{ij}\sim N\left[0,\sigma_\beta^2\right],\ \forall i,j;\ \varepsilon_{ijk}\sim N[0,\sigma_\varepsilon^2\equiv\sigma^2],\ \forall i,j,k;\tag{3}$$

$$\mathbf{C}\left[\alpha_i,\beta_{ij}\right]=\mathbf{C}\left[\alpha_i,\varepsilon_{ijk}\right]=\mathbf{C}\left[\beta_{ij},\varepsilon_{ijk}\right]=0,\ \forall i,j,k;\mathbf{C}\left[\varepsilon_{ijk},\varepsilon_{i'j'k'}\right]=0,\forall k\neq k',\tag{4}$$

$$Y_{ijk}\sim N[\mu,\sigma_Y^2],\ \forall ijk,$$

$$\sigma_{Y_{ijk}}^2\equiv\sigma_Y^2=\sigma_\alpha^2+\sigma_\beta^2+\sigma^2,\ \forall ijk,\tag{5}$$

where σ_α^2, σ_β^2 and σ^2 are *variance components* of measurement variance σ_Y^2 and $\mathbf{C}[u,v]$ is the covariance operator between u and v.

Here $\alpha_i+\beta_{ij}+\varepsilon_{ijk}=\Delta_{ijk}$ is the *true error* of Y_{ijk}.

3. PERG2FH ANOVA model of the main variance components for GPS measurements

Let for α and for β intervals of constant measurements be, i. e. let it be $I_{\alpha,1}=\cdots=I_{\alpha,i}\equiv I_\alpha$, and $I_{\beta,1}=\cdots=I_{\beta,j}\equiv I_\beta$, which means that the numbers of measurements in the intervals are equal to $b_{i1}=\cdots=b_{ij}\equiv b$ and $a_1=\cdots=a_i\equiv a$.

Applying the PERG2FH ANOVA model VC to model (1) one obtains the estimates of the varianc ecomponents (formulae (24), (25) and (26) in: [13]):

σ_α^2, σ_β^2 and σ^2.

Then the *mean value variance* $\overline{Y}=\Sigma\Sigma\Sigma Y_{ijk}/N$ ($N=abn$) in the hierarchical model will be:

$$\sigma_{\overline{Y}}^2=\frac{1}{a}\sigma_\alpha^2+\frac{1}{ab}\sigma_\beta^2+\frac{1}{abn}\sigma^2,\tag{6}$$

so that this expression will serve for the purpose of projecting, i. e. optimisation, of the measurement accuracy.

In the static method of precise phase measurement concerning GPS base vectors the measurement session lasts more than 10 minutes, so that $n\geq 20$ and, in this way also $N=abn\geq 20$. So, since in (6) the term swith σ_α^2 and σ_β^2 are not dominant, and when the number of measurements is increased, the third term decreases, i. e.:

$$(abn\nearrow)\Longrightarrow(\sigma^2/abn)\rightarrow 0,$$

so that in the further analysis insteadof (6) it is sufficient to use:

$$\sigma_{\overline{Y}}^2=\frac{1}{a}\sigma_\alpha^2+\frac{1}{ab}\sigma_\beta^2.\tag{7}$$

Thus, the accuracy $\overline{Y}$ depends on the tropo-iono variations σ_α and on the influence of quasi-stationary refraction blocks σ_β.

Note 1. In his papers Perović ([13], [14]) indicates existence of the quasi-stationary atmospheric refraction blocks known since long ago and, also on examples of measuring four base vectors 2, 20, 40 and 80 km long, confirms their existence, determining then the variance component σ_{β}^{2} for each coordinate difference, $\Delta\lambda$, $\Delta\varphi$, and Δh of the base vectors. However, many authors (to mention [7], [8] and [15] only) seem not to be informed of existence of quasi-stationary atmospheric refraction blocks, so that they the influences of these blocks a scribe to multipath. ▲

4. Optimising processes of GPS measurements

The GPS measurements can be performed in two ways.

The first one is to measure continuously $N(N = abn)$ times, as in the application of the two-factor hierarchical classifications, in which case for the purpose of estimating the mean value accuracy for the measured quantity (6), i. e. (7), would be used. However, here a problem appears of how large the numbers of intervals a and b are, especially because the duration of the quasi-stationary block is variable, it is between ~ 10minutes for a sunnyday – confirmed by the author's own experience, even as long as 3 hours by night [21]. If a high measurement accuracy is required, then the measuring process must be long, and consequently also expensive. Besides, in application of the measuring process we cannot know the duration of any measuring session so that for the measured quantity the accuracy remains unknown. Therefore, the author proposes application of an alternative way – with several sufficiently short measuring sessions.

The other way would comprise the performing of$s$short-lasting measurement sessions. For the sake of this it is to be reminded that in Introduction it was emphasized that $I_{\beta} \lesssim 2$ h, whereas $I_{\alpha} \gtrsim 3$ h. These facts and the analysis of expression (7) enable us to optimise the measuring process. In this way the influences of the quasi-stationary refractionb locks will be randomised if the measuring sessions are performed, generally, for $I_{\beta} \lesssim 2$ h, whereas the tropo-iono influences will be randomised for $I_{\alpha} \gtrsim 3$ h .

So, let *the measuring duration in a session*, T_{S}, be short – say $T_{\mathrm{S}} \lesssim 1$ h, and if for the time interval *between (two consecutive) sessions* $B_{\mathrm{S}} \gtrsim 3$ h is assumed, *the number of sessions* denoted as s (now within a session it wil be $I_{\beta} = I_{\alpha}$), we shall have $a = s$ and $ab = s$, then based on (7) we obtain *the expression for estimating the accuracy of* GPS measurements:

$$\sigma_{\bar{Y}}^{2} = \frac{1}{s}\sigma_{\alpha}^{2} + \frac{1}{s}\sigma_{\beta}^{2}, \qquad \text{– for } s \text{ sessions, } (B_{S} \gtrsim 3 \text{ h}, T_{\mathrm{S}} \lesssim 1 \text{ h}). \tag{8}$$

Here it is:

$$2 \leq efficient\ number\ of\ sessions\ \ s \leq 4, \tag{9}$$

where, because of reliability of measurements, it must be:

$$\min s = 2. \tag{10}$$

5. Determining VC projected measurements

The question of how to determine σ_{α} and σ_{β} for the foreseen, i. e. projected, GPS measurements remains. For this purpose detailed studies should be done and, until that occurs, in approximate computations the author's determinations of the variance components σ_{α}^{2} and σ_{β}^{2} [13], [14] for the four base vectors, 2, 20, 40 and 80 km long, can be used.

Working on this problem the author has inferred that by means of the following family of models:

$$a + b\sqrt{D} \tag{11}$$

σ_α and σ_β can be well described, where the units are: for a – mm and, for b – $\mathrm{mm}/\sqrt{\mathrm{km}}$. On the basis of the measurement results for the mentioned vectors [13], [14] by applying least squares it is obtained:

$$\left.\begin{aligned} \sigma_{\alpha,\Delta\lambda} &= 1.39 + 0.20\sqrt{D} \\ \sigma_{\alpha,\Delta\varphi} &= 1.49 + 0.33\sqrt{D} \\ \sigma_{\alpha,\Delta h} &= 2.29 + 1.56\sqrt{D} \end{aligned}\right\}, \tag{12a}$$

$$\left.\begin{aligned} \sigma_{\beta,\Delta\lambda} &= 1.40 + 0.43\sqrt{D} \\ \sigma_{\beta,\Delta\varphi} &= 3.00 + 0.36\sqrt{D} \\ \sigma_{\beta,\Delta h} &= 5.66 + 0.86\sqrt{D} \end{aligned}\right\}. \tag{12b}$$

In this way, for instance, for the measuring of the planned base GPS vector of intensity $D = 5$ km in *one session* it would be obtained:

$$D = 5\ \mathrm{km}_{\,s=1\ \mathrm{session}}:\quad \sigma_{\alpha,\Delta\lambda} = 1.8\ \mathrm{mm},\ \ \sigma_{\alpha,\Delta\varphi} = 2.2\ \mathrm{mm},\ \ \sigma_{\alpha,\Delta h} = 5.8\ \mathrm{mm},$$

$$D = 5\ \mathrm{km}_{\,s=1\ \mathrm{session}}:\quad \sigma_{\beta,\Delta\lambda} = 2.4\ \mathrm{mm},\ \ \sigma_{\beta,\Delta\varphi} = 3.8\ \mathrm{mm},\ \ \sigma_{\beta,\Delta h} = 7.6\ \mathrm{mm},$$

whereas based on *two sessions* ($s = 2$) it would be obtained:

$$D = 5\ \mathrm{km}_{\,s=2\ \mathrm{sessions}}:\quad \sigma_{\alpha,\Delta\lambda} = 1.3\ \mathrm{mm},\ \ \sigma_{\alpha,\Delta\varphi} = 1.6\ \mathrm{mm},\ \ \sigma_{\alpha,\Delta h} = 4.1\ \mathrm{mm},$$

$$D = 5\ \mathrm{km}_{\,s=2\ \mathrm{sessions}}:\quad \sigma_{\beta,\Delta\lambda} = 1.7\ \mathrm{mm},\ \ \sigma_{\beta,\Delta\varphi} = 2.7\ \mathrm{mm},\ \ \sigma_{\beta,\Delta h} = 5.4\ \mathrm{mm},$$

thus improved by 30%. For $s = 3$ we would have an improvement of 40%, for $s = 4$ – 50%.

Note 2. *From expressions* (*12a*) *and* (*12b*) *approximate values of the standards are obtained, but they are sufficiently good, to be used in formation of the projects concerning geodetic control networks*. ▲

6. Conclusions

As for the presented material concerning the main variance components of the GPS measurements the following conclusions can be formulated:

C1) The variance of the GPS measurements can be described by means of three structural components: 1) tropo-iono variations (α), 2) influence of quasi-stationary atmospheric refraction blocks (β) and 3) purely random deviations of the GPS measurements (ε);

C2) since in the applications a large number of measurements concerning a single quantity is used, the variance of the GPS measurements can be sufficiently well described by using the first two main variance components, σ_α^2 and σ_β^2, i. e. by using model family (11), $a + b\sqrt{D}$,

C3) in order to be possible to use the model $a + b\sqrt{D}$ in practice the model parameters a and b should be better examined, as it is done in example (12*a*-*b*),

C4) by applying model (8), i. e. (11), for determination of VC projected measurements, a significant reduction of observing time is achieved, i. e. the price of the works is significantly reduced.

C5) conclusions C2) and C3) are valid for the base vectors of magnitude by 80 km, whereas for those the magnitude of which exceeds 80 km the validity should be verified.

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